

EXISTENCE OF STANDING WAVES FOR THE COMPLEX GINZBURG-LANDAU EQUATION

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ABSTRACT. We prove the existence of non-trivial standing wave solutions of the complex Ginzburg-Landau equation $\phi_t - e^{i\theta}(\rho I - \Delta)\phi - e^{i\gamma}|\phi|^\alpha\phi = 0$ in \mathbb{R}^N , where $(N-2)\alpha < 4$, $\theta, \gamma \in (-\pi/2, \pi/2)$ and $\rho > 0$. Analogous result is obtained in a ball $\Omega \subset \mathbb{R}^N$ for $\rho > -\lambda_1$, where λ_1 is the first eigenvalue of the Laplace operator with Dirichlet boundary conditions.

1. INTRODUCTION

The complex Ginzburg-Landau equation

$$\psi_t = z_1 \Delta \psi + z_2 |\psi|^\alpha \psi + z_3 \psi, \quad (1.1)$$

for $\alpha = 2$, $z_1, z_2, z_3 \in \mathbb{C}$, with $\Re z_1 \geq 0$ was proposed independently by DiPrima, Eckhaus, Segel [11] and Stewartson, Stuart [33] to model the interaction of plane waves in fluid flows and plays a central role in the study of the development of nonlinear instabilities in fluid dynamics. See [13, 7, 35] and the references cited therein for a discussion of various problems where the complex Ginzburg-Landau equation applies. Local (global for $\Re z_2 < 0$) well-posedness of (1.1) (for $\alpha > 0$) were derived in both \mathbb{R}^N and a domain $\Omega \subset \mathbb{R}^N$, under various boundary conditions and assumptions on the parameters, in [14, 15, 16, 21, 22, 25, 27, 28, 29, 30].

2010 *Mathematics Subject Classification.* 35Q56, 35C08.

Key words and phrases. Standing waves, complex Ginzburg-Landau equation.

Flávio Dickstein was partially supported by CNPq (Brasil).

This work has been done while Jean-Pierre Puel was visiting Universidade Federal do Rio de Janeiro as a "Professor Visitante Especial" of the "Programa Ciência sem Fronteiras" of Capes/CNPq (Brasil).

The existence of special solutions of (1.1) (holes, fronts, pulses, sources, sinks, etc) is discussed in numerous works, see e.g. [6, 8, 10, 12, 19, 20, 24, 26, 31, 32, 35]. We look for standing wave solutions. Replacing φ by $e^{i\eta t}\varphi$ for some $\eta \in \mathbb{R}$ and rescaling the equation, we rewrite (1.1) as

$$\partial_t \varphi + e^{i\theta}(\rho\varphi - \Delta\varphi) = e^{i\gamma}|\varphi|^\alpha\varphi, \quad (1.2)$$

where $\rho \in \mathbb{R}$. Given $\omega \in \mathbb{R}$, a standing wave of the form $\varphi = e^{i\omega t}u(x)$ is a solution of (1.2) if and only if u satisfies

$$i\omega u + e^{i\theta}(\rho u - \Delta u) = e^{i\gamma}|u|^\alpha u. \quad (1.3)$$

Plane waves $\varphi = e^{i(kx - \omega t)}$, where $k, \omega \in \mathbb{R}$ are particular standing waves. It is easy to see that (1.2) admits plane wave solutions in \mathbb{R}^N for all values of ρ , θ , γ and α . Stationary solutions are also standing waves of special kind. If $\omega = 0$ and $u \neq 0$ then necessarily $\sin \gamma = \sin \theta$, so that equation (1.3) reduces to the nonlinear elliptic equation $\rho u - \Delta u = \pm |u|^\alpha u$. The case of the nonlinear Schrödinger equation $\theta = \pm \gamma = \pm \frac{\pi}{2}$ leads to the equation $\Delta u + |u|^\alpha u - (\rho \mp \omega)u = 0$.

We will obtain solutions that are different from these particular ones. In fact, using well known results of the theory of nonlinear elliptic equations for the case $\omega = 0$ and $\theta = \gamma$, we show the existence of nontrivial standing wave solutions for $\theta \neq \gamma$ by a perturbation argument, as we describe below.

Equation (1.3) will be considered both in the whole space $\Omega = \mathbb{R}^N$ or in a ball $\Omega \in \mathbb{R}^N$ with Dirichlet boundary condition, for $N \geq 1$. We suppose $\theta, \gamma \in (-\pi/2, \pi/2)$ and α subcritical, i.e.

$$(N - 2)\alpha < 4, \quad (1.4)$$

which includes the relevant case $\alpha = 2$, for $N \leq 3$. For $\theta = \gamma$ and $\omega = 0$, (1.3) reduces to

$$\rho u - \Delta u - |u|^\alpha u = 0. \quad (1.5)$$

Consider first $\Omega = \mathbb{R}^N$, in which case we assume that $\rho > 0$. It is then known that (1.5) has a unique positive radially symmetric solution $U \in C^2(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$. In fact, $U \in H_{\text{rad}}^2(\mathbb{R}^N)$, the subspace of radial functions of $H^2(\mathbb{R}^N)$. Note that (1.5) is phase invariant, i.e., $Ue^{i\beta}$ is also a solution for all $\beta \in \mathbb{R}$. We prove the following result, in which the Hilbert spaces are real, but composed of complex-valued functions.

Theorem 1.1. *Assume (1.4) holds and suppose $\rho > 0$. Let $U \in H_{\text{rad}}^2(\mathbb{R}^N)$ be the unique positive radial solution of (1.5). Given $\theta \in (-\pi/2, \pi/2)$ and $\beta \in \mathbb{R}$ there exists $0 < \varepsilon < \min\{\pi/2 - \theta, \pi/2 + \theta\}$ and a C^1 mapping $g : (\theta - \varepsilon, \theta + \varepsilon) \rightarrow \mathbb{R} \times H_{\text{rad}}^2(\mathbb{R}^N)$, $g(\gamma) = (\omega_\gamma, u_\gamma)$, satisfying $\omega_\theta = 0$, $u_\theta = Ue^{i\beta}$ and such that $\varphi_\gamma = e^{i\omega_\gamma t}u_\gamma$ is a solution of (1.2).*

In the bounded domain case of the unitary ball Ω of \mathbb{R}^N , we suppose that

$$\rho > -\lambda_1, \quad (1.6)$$

where λ_1 is the first eigenvalue associate to the Laplace-Dirichlet operator in Ω . As in the case of the whole space, (1.5) admits a unique positive solution $U \in H^2(\Omega) \cap H_0^1(\Omega)$, which is radial and radially decreasing. The following result is analogous to Theorem 1.1.

Theorem 1.2. *Assume (1.4), (1.6) hold and let $U \in H^2(\Omega) \cap H_0^1(\Omega)$ be the positive solution of (1.5). Given $\theta \in (-\pi/2, \pi/2)$ and $\beta \in \mathbb{R}$ there exists $0 < \varepsilon < \min\{\pi/2 - \theta, \pi/2 + \theta\}$ and a C^1 mapping $g : (\theta - \varepsilon, \theta + \varepsilon) \rightarrow \mathbb{R} \times (H^2(\Omega) \cap H_0^1(\Omega))$, $g(\gamma) = (\omega_\gamma, u_\gamma)$, satisfying $\omega_\theta = 0$, $u_\theta = Ue^{i\beta}$ and such that $\varphi_\gamma = e^{i\omega_\gamma t} u_\gamma$ is a solution of (1.2).*

In the proofs of Theorem 1.1 and Theorem 1.2 we apply the Implicit Function Theorem to $F(w, u, \gamma) = i\omega u + e^{i\theta}(\rho u - \Delta u) - e^{i\gamma}|u|^\alpha u$ in a neighborhood of $w = 0$, $u = Ue^{i\beta}$ and $\gamma = \theta$. Analogous approach was considered in [4] to obtain standing wave solutions to (1.2) in a bounded domain for α small, where an eigenvector of the Laplace-Dirichlet operator is used as a starting point. Our point of view allows us to obtain solutions for α satisfying (1.4) and for the case of the whole space. We are lead to study the linearized operator $L_\beta = \partial_u F(0, Ue^{i\beta}, \theta)$ in an appropriate setting. In fact, it will be sufficient to consider $L = \partial_u F(0, U, \theta)$, see Section 5.

We address some comments about the hypothesis in Theorem 1.1 and Theorem 1.2.

- Remark 1.3.**
- (1) The assumption $\theta \in (-\pi/2, \pi/2)$ yields an accretive linear operator associated to the problem and corresponds to $\Re z_1 > 0$ in (1.1). We also obtain $\gamma \in (-\pi/2, \pi/2)$, i.e., standing waves appear in the focusing case. In the defocusing case $\gamma \in (\pi/2, 3\pi/2)$, multiplying the equation by $\bar{\varphi}$ and integrating, we see that $\|\varphi(t)\|_{L^2(\mathbb{R}^N)}$ decreases in time. Thus there cannot be any non-trivial standing wave in that case.
 - (2) The restriction to radial solutions in Theorem 1.1 seems to be necessary in our proof. It ensures the compactness of the linear operator K introduced in the proof. It also ensures that $\ker L$ is one-dimensional, which allows for the application of the Implicit Function Theorem. As discussed in Section 3, $\ker L$ is $(N + 1)$ -dimensional in $L^2(\mathbb{R}^N)$.
 - (3) The assumption that Ω is a ball in Theorem 1.2 ensures that $\ker L$ is one-dimensional. We don't know if this is true in general. The standing waves in Theorem 1.2 can be constructed such that they are radially symmetric, see Remark 5.1.

This paper is organized as follows. In Section 2 we recall some well established properties of the positive solution U , both in the bounded and in the unbounded domain cases. A spectral analysis of the operator L is developed in Section 4 for the case where Ω is a ball, and in Section 3 when Ω is the whole space. Finally, in Section 5 we prove Theorem 1.1 and Theorem 1.2.

2. THE STARTING POINT $\theta = \gamma$

In this section we recall some well known properties of solutions $u \in H_0^1(\Omega)$ of (1.5) which will be useful later.

We consider first the case where Ω is a ball and we assume (1.6). Then (1.5) admits infinitely many real solutions and, in particular, a positive radially symmetric solution U [1]. Equation (1.5) is phase invariant: if u solves (1.5) so does $e^{i\beta}u$ for all $\beta \in \mathbb{R}$. Non-radial complex solutions in \mathbb{R}^N were obtained in [23].

The positive solution U , which was shown to be unique in [18], can be obtained by ode's methods [3]. It can also be derived by solving the minimization problem

$$\min_{u \in S} \int \rho |u|^2 + |\nabla u|^2, \quad (2.1)$$

where

$$S = \{u \in H_0^1(\Omega), \int |u|^{\alpha+2} = 1\}. \quad (2.2)$$

Using that $H^1(\Omega)$ is compactly injected in $L^{\alpha+2}(\Omega)$ one easily sees that (2.1) has a (unique) positive solution \tilde{U} . It is also clear that $U = k\tilde{U}$ solves (1.5) for a judicious choice of k . It then follows from standard symmetrization arguments that U is radial and radially decreasing.

One may also obtain U as a mountain pass solution. Consider

$$E(u) = \frac{\rho}{2} \int |u|^2 + \frac{1}{2} \int |\nabla u|^2 - \frac{1}{\alpha+2} \int |u|^{\alpha+2}. \quad (2.3)$$

and

$$\Gamma = \{\gamma \in C([0, 1]; H_0^1(\Omega)), \gamma(0) = 0, \gamma(1) = u_1\}, \quad (2.4)$$

where $E(u_1) < 0$. Then $E(u)$ is well-defined for $u \in H_0^1(\Omega)$ and Γ is nonempty. In addition,

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} E(\gamma(t)) \quad (2.5)$$

is a critical value of E such that $c = E(U) > 0$ and $E'(U) = 0$. Moreover, it can be easily shown that U is a ground state solution, i.e., $E(U) \leq E(V)$ for all solution $V \neq 0$ of (1.5).

The general picture essentially remains unchanged for real solutions u of (1.5) in the whole space \mathbb{R}^N , provided $\rho > 0$ and $u \in H_{\text{rad}}^1(\mathbb{R}^N)$, the space of radially symmetric functions of $H^1(\mathbb{R}^N)$. In fact, using that $H_{\text{rad}}^1(\mathbb{R}^N)$ is compactly injected in $L^{\alpha+2}(\mathbb{R}^N)$ [34], the existence of a positive solution U can be obtained either by solving (2.1) or (2.5), when S and Γ are redefined by replacing $H_0^1(\Omega)$ by $H_{\text{rad}}^1(\mathbb{R}^N)$, see (2.2), (2.4). Again, symmetrization arguments ensure that U is radially decreasing. In fact, U decays exponentially [2]. It is not difficult to see that both methods provide the same solution. However, in [18] it is also shown the uniqueness of positive radially symmetric solutions in the whole space. (An alternative variational characterization of U involving the so-called Gagliardo-Nirenberg quotient is presented in [36, Proposition 2.6].

For Ω either the unitary ball or the whole space, we consider the linearized operator

$$Lv = \rho v - \Delta v - U^\alpha v - \alpha U^\alpha \Re v, \quad (2.6)$$

where U is the positive solution of (1.5). More precisely, we set $D(L) = H^2(\Omega) \cap H_0^1(\Omega)$ and define $L : D(L) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ by (2.6). Then $Lv = L_+ \Re v + iL_- \Im v$ where

$$L_+ v = \rho v - \Delta v - (\alpha + 1)|U|^\alpha v, \quad (2.7)$$

$$L_- v = \rho v - \Delta v - |U|^\alpha v. \quad (2.8)$$

We study below the operators L_+ and L_- .

3. THE LINEARIZED OPERATOR: THE CASE OF $\Omega = \mathbb{R}^N$

In the case $\Omega = \mathbb{R}^N$, under a suitable rescaling we may assume that $\rho = 1$ in (2.6). We want to show that L_+ is an injective operator when restricted to the space $V := L_{\text{rad}}^2(\mathbb{R}^N)$ of radially symmetric and square integrable functions. We define $D(L_+) = H^2(\mathbb{R}^N) \cap V$ and consider $L_+ : D(L_+) \subset V \rightarrow V$ given by (2.7).

Set $\sigma = U^\alpha$ and denote V_σ the space $L^2(\mathbb{R}^N)$ equipped with the scalar product

$$\langle u, v \rangle_\sigma = \int \sigma uv. \quad (3.1)$$

We also introduce $K : V_\sigma \rightarrow V_\sigma$ such that for $v \in V_\sigma$

$$Kv = (\alpha + 1)(I - \Delta)^{-1}U^\alpha v. \quad (3.2)$$

We have that K is a positive, symmetric operator. Using that U decays to zero at infinity, a standard argument shows that K is compact. Denote $\{\varphi_j\}_{j \in \mathbb{N}}$ the orthonormal basis of eigenvectors of K and $\{\lambda_j\}_{j \in \mathbb{N}}$ the corresponding set of eigenvalues. Then $\lambda_j > 0$ and $K\varphi_j = \lambda_j\varphi_j$ is equivalent to

$$\varphi_j - \Delta\varphi_j = \frac{\alpha + 1}{\lambda_j}U^\alpha\varphi_j. \quad (3.3)$$

Therefore $\varphi_1 = U$ and $\lambda_1 = \alpha + 1$. We will now prove that $\lambda_2 \leq 1$. This is a consequence of the fact that U is a mountain-pass solution. We present a simple proof below, which uses the specific form of the function $E(u)$. For the proof that general critical points of mountain-pass type have Morse index equal to one, see [17]

Lemma 3.1. $\lambda_2 \leq 1$.

Proof. We first remark that for $k > 1$ large enough $\gamma_0(t) = ktU \in \Gamma$, see (2.4). In addition, $\max_{u \in \gamma_0} E(u) = E(U)$.

We argue by contradiction and suppose that $\lambda_2 > 1$. We get from (3.3) that

$$\langle L_+\varphi_2, \varphi_2 \rangle = (\alpha + 1)(\lambda_2^{-1} - 1) \int U^\alpha \varphi_2^2 < 0. \quad (3.4)$$

Consider now the plane π containing U and φ_2 and let

$$\gamma_1 = \{U - \delta(\cos tU + \sin t\varphi_2), t \in [0, \pi]\}. \quad (3.5)$$

be an arc of circle in π joining $(1 - \delta)U$ and $(1 + \delta)U$. We have that

$$E(u) = E(U) + E'(U)(u - U) + \frac{1}{2}\langle L_+(u - U), u - U \rangle + o(\delta^2). \quad (3.6)$$

Moreover, using that $\langle L_+U, \varphi_2 \rangle = -\alpha\langle U, \varphi_2 \rangle_\sigma = 0$ we get

$$\langle L_+(u - U), u - U \rangle = \delta^2(\cos^2 t \langle L_+U, U \rangle + \sin^2 t \langle L_+\varphi_2, \varphi_2 \rangle) < 0. \quad (3.7)$$

Using this, (3.6) and that $E'(U) = 0$, we see that we can choose δ small enough so that $E(u) < E(U)$ for all $u \in \gamma_1$.

Let now γ be the curve obtained by replacing the path of γ_0 going from $(1 - \delta)U$ to $(1 + \delta)U$ by γ_1 . Then $\gamma \in \Gamma$ and $\max_{u \in \gamma} E(u) < E(U)$, leading to a contradiction. This shows that $\lambda_2 \leq 1$. \square

Lemma 3.2. Suppose $L_+\varphi = 0$, $\varphi \neq 0$. Then there exists a unique $r^* > 0$ such that $\varphi(r^*) = 0$.

Proof. Let B_R be the ball of radius R of \mathbb{R}^N . For $v \in L_{\text{rad}}^2(B_R)$ let $u \in H_{\text{rad}}^2(B_R) \cap H_0^1(B_R)$ satisfy $(I - \Delta)u = (\alpha + 1)U^\alpha v$. We define $K_R : L_{\text{rad}}^2(B_R) \rightarrow L_{\text{rad}}^2(B_R)$ such that $K^R v = u$. Then K^R is a compact operator, which is symmetric and positive for the scalar product

$$\langle u, v \rangle_{\sigma, R} = \int_{B_R} U^\alpha uv. \quad (3.8)$$

Denote $\{\varphi_j^R\}_{j \in \mathbb{N}}$ an orthonormal basis of eigenvectors of K^R , associate to the set $\{\lambda_j^R\}_{j \in \mathbb{N}}$ of eigenvalues, so that

$$(I - \Delta)\varphi_j^R = \frac{\alpha + 1}{\lambda_j^R} U^\alpha \varphi_j^R \quad (3.9)$$

and

$$\int_{B_R} U^\alpha \varphi_i^R \varphi_j^R = \delta_{ij}. \quad (3.10)$$

Moreover, it is easy to see that

$$\lambda_j^R < \lambda_1^R < \lambda_1, \quad (3.11)$$

where λ_1 is the first eigenvalues of K given by (3.2), and that for all j

$$\lambda_j^R \nearrow \lambda_j^\infty \leq \lambda_1 \quad (3.12)$$

as $R \rightarrow \infty$. We extend $\varphi_j^R(r) = 0$ for $r > R$. Using (3.9) and (3.10) we see that

$$\int |\varphi_j^R|^2 + |\nabla \varphi_j^R|^2 = \frac{\alpha + 1}{\lambda_j^R} \int_{B_R} U^\alpha |\varphi_j^R|^2 = \frac{\alpha + 1}{\lambda_j^R}. \quad (3.13)$$

It follows then from (3.12) that $\{\varphi_j^R\}_{R \geq \underline{R}}$ is uniformly bounded in $H^1(\mathbb{R}^N)$ for all $\underline{R} > 0$. Upon considering a subsequence, we may write that there exists φ^∞ in $H^1(\mathbb{R}^N)$ such that $\varphi_j^R \rightharpoonup \varphi_j^\infty$ weakly in $H^1(\mathbb{R}^N)$ as $R \rightarrow \infty$. Using that $U(r) \rightarrow 0$ as $r \rightarrow \infty$, we readily obtain that

$$\int_{B_R} U^\alpha \varphi_i^\infty \varphi_j^\infty = \delta_{ij}, \quad (3.14)$$

with

$$(I - \Delta)\varphi_j^\infty = \frac{\alpha + 1}{\lambda_j^\infty} U^\alpha \varphi_j^\infty. \quad (3.15)$$

Thus, λ_j^∞ is an eigenvalue of K , associated to φ_j^∞ .

Suppose now that $L_+\varphi = 0$ so that $K\varphi = \varphi$. Assume that $\varphi(\rho) = \varphi(R) = 0$ for some $0 < \rho < R$. Then $K^R\varphi = \varphi$. Thus $1 = \lambda_j^R$ for some j . However, since φ changes sign once in $(0, R)$, $1 = \lambda_2^R$. Thus $\lambda_2^\infty > 1$. Since $\lambda_2^\infty \neq \lambda_1$, we see that $1 < \lambda_2^\infty \leq \lambda_2$. But this contradicts Lemma 3.1. Thus, $\varphi(r)$ has a single zero $r^* > 0$. \square

We next present the ingenious argument of [5] to show that L_+ is injective.

Lemma 3.3. *L_+ is injective.*

Proof. We argue by contradiction and assume that there exists $\varphi \neq 0$ such that $L_+\varphi = 0$. Using Lemma 3.2, we may assume that there exists $r^* > 0$ such that $\varphi(r) > 0$ for $r < r^*$ and $\varphi(r) < 0$ for $r > r^*$. Set now

$$\eta(x) = U(x) + \frac{\alpha}{2} x \cdot \nabla U(x) \quad (3.16)$$

Since U decays exponentially, $\eta \in H_{\text{rad}}^1(\mathbb{R}^N)$. Moreover, a straightforward calculation yields

$$L_+\eta = -\alpha U. \quad (3.17)$$

Define $w = U^\alpha(r^*)\eta - U$ and $z = L_+w$. Then $z(r) = \alpha U(r)(U^\alpha(r) - U^\alpha(r^*))$ and so $z(r) > 0$ for $r < r^*$ and $z(r) < 0$ for $r > r^*$. Hence, $z(r)\varphi(r) > 0$ for $r \neq r^*$. However, this is in contradiction with the fact that

$$\langle \varphi_2, z \rangle = \langle \varphi_2, L_+w \rangle = \langle L_+\varphi_2, w \rangle = 0.$$

This shows that L_+ is injective. \square

Using decomposition in spherical harmonics, in [36] and in [5] it is proved that the complete kernel of L_+ in $L^2(\mathbb{R}^N)$ is $\ker L_+ = [\partial_1 U, \partial_2 U, \cdot, \partial_N U]$. Note that $\partial_j U$ is not a radial function.

We may now characterize the kernel of L given by (2.6).

Proposition 3.4. *We have $\ker L = [iU]$.*

Proof. If $v \in \ker L$ then $\Re v \in \ker L_+$ and $\Im v \in \ker L_-$. It follows from Lemma 3.3 that $\Re v = 0$. Moreover, if $\varphi \in \ker L_-$ then φ is an eigenvalue of K given by (3.2), associated to $\lambda = \alpha + 1$. But $\alpha + 1$ is the first eigenvalue of K and $KU = (\alpha + 1)U$. Hence $\ker L_- = [U]$ so that $\ker L = [iU]$. \square

4. THE LINEARIZED OPERATOR: THE CASE OF A BALL

Let $\Omega \subset \mathbb{R}^N$ be the unitary ball and suppose (1.6) holds. Let U be the unique positive solution of (1.5) and let L be given by (2.6). Then $v \in \ker L$ if and only if $\Re v \in \ker L_+$ and $\Im v \in \ker L_-$. Since $L_-U = 0$ and $U > 0$, it follows that $\ker L_- = [U]$ is a one-dimensional subspace. We will now show that L_+ is injective. This is proved in [9] for $\rho = 0$. For the reader's convenience, we reproduce the arguments here. The two preliminary results, Lemma 4.1 and Lemma 4.1 hold in fact for $\rho > -\lambda_1$ and will be useful in the proof of the general case.

For $x \in \mathbb{R}^N$ write $x = (t, y)$, where $t \in \mathbb{R}$, $y \in \mathbb{R}^{N-1}$, if $N > 1$ or $x = t$ if $N = 1$. Set $\Omega^* = \{x \in \Omega, t < 0\}$, $D(L_+^*) = H^2(\Omega^*) \cap H_0^1(\Omega^*)$ and $L_+^* : D(L_+^*) \subset L^2(\Omega^*) \rightarrow L^2(\Omega^*)$ be given by (2.7).

Lemma 4.1. *We have $\lambda_1^* = \lambda_1(L_+^*) > 0$.*

Proof. Let $v = \partial_t U$. It is well known that $v > 0$ over Ω^* with $v > 0$ over $\Gamma^* = \{x \in \overline{\Omega^*}, |x| = 1\}$. Moreover, taking the derivative with respect to t in (1.5) we see that $L_+^*v = 0$. Consider u_1 a positive eigenvector of L_+^* , so that $L_+^*u_1 = \lambda_1^*u_1$. Then

$$\lambda_1^* \int_{\Omega^*} u_1 v = \int_{\Omega^*} v L_+^* u_1 = - \int_{\partial\Omega^*} v \partial_\eta u_1 > 0. \quad (4.1)$$

This shows that $\lambda_1^* > 0$. \square

As a consequence, we have the following.

Lemma 4.2. *Let v satisfy $L_+v = 0$. Then v is radially symmetric.*

Proof. If $v \in \ker L_+$ then $v \circ R \in \ker L_+$ for all unitary transformation R . It thus suffices to show that $v(t, y)$ is symmetric with respect to t . Define $\psi(x) = v(t, y) - v(-t, y)$. Then $L_+^*\psi = 0$, with $\psi = 0$ over $\partial\Omega^*$. It follows from Lemma 4.1 that $\psi = 0$. This ends the proof. \square

Lemma 4.3. *Suppose $\rho = 0$. Then the operator L_+ given by (2.8) is injective.*

Proof. Let $v \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfy $L_+v = 0$. Then

$$0 = \int_{\Omega} L_+v U = \int_{\Omega} L_+U v = -\alpha \int_{\Omega} U^{\alpha+1}v. \quad (4.2)$$

Consider now the Pohozaev function $\psi = x \cdot \nabla U$. We have that $\partial_j \psi = \partial_j U + x \cdot \nabla \partial_j U$, so that $\partial_{jj}^2 \psi = 2\partial_{jj}^2 U + x \cdot \nabla \partial_{jj}^2 U$. Thus, $\Delta \psi = 2\Delta U + x \cdot \nabla \Delta U$. In addition, we get from (1.5) that $\Delta \nabla U + (\alpha + 1)U^\alpha \nabla U = 0$, and so

$$\Delta \psi = 2\Delta U - (\alpha + 1)U^\alpha x \cdot \nabla U = -2U^{\alpha+1} - (\alpha + 1)U^\alpha \psi. \quad (4.3)$$

It follows from (4.3) and (4.2) that

$$0 = 2 \int_{\Omega} U^{\alpha+1}v = \int_{\Omega} L_+ \psi v = \int_{\partial\Omega} \psi \partial_\eta v = \int_{\partial\Omega} \partial_\eta U \partial_\eta v. \quad (4.4)$$

We conclude from Lemma 4.2 and Hopf's strong maximum principle that $v'(1) = 0$. Therefore, $v = 0$. \square

Remark 4.4. Following [9], Lemma 4.3 holds in the case $N = 2$ and Ω any regular bounded star shaped domain.

The proof that L_+ injective in the case $\rho \neq 0$ follows the arguments of [5], see Lemma 3.3.

Lemma 4.5. *Assume $\rho > -\lambda_1$, $\rho \neq 0$. Then L_+ is injective.*

Proof. Let

$$\eta(r) = 2U(r) + \alpha r U'(r). \quad (4.5)$$

A straightforward calculation gives that

$$L_+ \eta = -2\rho \alpha U. \quad (4.6)$$

We want to show that $\ker L_+ = \{0\}$. We argue by contradiction and assume that there exists $\varphi_2 \neq 0$ such that $L_+ \varphi_2 = 0$. Since U is a mountain-pass solution of (1.5), we know that $\lambda_2(L_+) \geq 0$, see [17]. Thus, φ_2 is an eigenvector associated to the second eigenvalue $\lambda_2 = 0$. By Lemma 4.2, φ_2 is radial. Using standard comparison arguments, it is easy to see that φ_2 has a single zero r_0 in $(0, 1)$. For $b \in (0, 1)$, define

$$g(r) = \begin{cases} 1 & 0 \leq r < b, \\ 2\frac{1-r}{1-b} - \frac{(1-r)^2}{(1-b)^2} & b \leq r \leq 1. \end{cases} \quad (4.7)$$

Then $g''(r) = g'(r) = 0$ if $r < b$. For $r > b$,

$$g'(r) = \frac{2(b-r)}{(1-b)^2}, \quad (4.8)$$

$$g''(r) = -\frac{2}{(1-b)^2}. \quad (4.9)$$

We remark that we can choose b close enough to 1 so that

$$b > r_0 \quad (4.10)$$

and

$$U^\alpha(r) < U^\alpha(r_0)g(r) \quad (4.11)$$

for $r > b$. Set now $w(r) = g(r)\eta(r)$, see (4.5). It follows from (4.6) that

$$L_+w = gL_+\eta - 2\nabla g \cdot \nabla \eta - \eta \Delta g = -2g\rho\alpha U - 2\nabla g \cdot \nabla \eta - \eta \Delta g. \quad (4.12)$$

Thus $L_+w = -2\rho\alpha U$ if $|x| < b$.

For $|x| > b$, we get from (4.5), (4.8) and (4.8) that

$$\nabla g(r) \cdot \nabla \eta(r) = g'(r) \eta'(r) = \frac{2(b-r)}{(1-b)^2} ((2+\alpha)U'(r) + \alpha r U''(r)) \quad (4.13)$$

and that

$$\Delta g = g'' + \frac{N-1}{r} g' = -\frac{2}{(1-b)^2} - \frac{2(N-1)(r-b)}{r(1-b)^2}. \quad (4.14)$$

Defining $h = -2\nabla g \cdot \nabla \eta - \eta \Delta g$ we get from (4.5), (4.13) and (4.14) that there exists $K > 0$ such that

$$(1-b)^2 h(r) \leq 4U(r) + 2\alpha r U'(r) + K(r-b).$$

Since $U(1) = 0$ and $U'(1) < 0$, $1-b$ can be taken eventually smaller so that

$$h(r) < 0 \text{ for } r > b. \quad (4.15)$$

Set now $t = U^\alpha(r_0)/(2\rho)$ and $z = L_+(-U + tw)$. Hence, by (4.12) we get

$$z = \alpha U^{\alpha+1} + t(-2g\rho\alpha U + h). \quad (4.16)$$

Let us show that z and φ_2 have the same sign. For $r < b$ we use that $g = 1$ and $h = 0$ to get

$$z(r) = \alpha U(r)(U^\alpha(r) - U^\alpha(r_0)).$$

It follows that $z(r) > 0$ if $r < r_0$ and $z(r) < 0$ for $r \in (r_0, b)$. In addition, using (4.15) and (4.11), we get for $r > b$ that

$$z(r) < \alpha U(r)(U^\alpha(r) - gU^\alpha(r_0)) < 0.$$

We see then that $z(r)\varphi_2(r) > 0$ for $r \neq r_0$. But

$$\langle \varphi_2, z \rangle = \langle \varphi_2, L_+(-U + \beta w) \rangle = \langle L_+\varphi_2, -U + \beta w \rangle = 0,$$

giving a contradiction. This shows that L_+ is injective. \square

We present now the main result of this section.

Proposition 4.6. *We have $\ker L = [iU]$.*

Proof. Let $v \in \ker L$. Then $\Re v \in \ker L_+$ and $\Im v \in \ker L_-$. It follows from Lemma 4.3 and Lemma 4.5 that $\Re v = 0$. Moreover, as discussed in the beginning of this section $\ker L_- = [U]$. This closes the proof. \square

5. PROOFS OF THEOREM 1.1 AND THEOREM 1.2

In this section we denote $L^p(\Omega)$ the real Banach space whose elements are complex-valued functions. In particular, $L^2(\Omega)$ is a Hilbert space for the scalar product

$$(u, v) = \Re \int_{\Omega} u \bar{v}. \quad (5.1)$$

Accordingly, $H^m(\Omega)$ denotes a real Hilbert space having complex elements.

Proof of Theorem 1.1. For a fixed $\theta \in (-\pi/2, \pi/2)$ set $X = \mathbb{R} \times (H_{\text{rad}}^2(\mathbb{R}^N))$ and $F : (-\pi/2, \pi/2) \times X \rightarrow L_{\text{rad}}^2(\mathbb{R}^N)$ such that

$$F(\gamma, \omega, u) = \rho u - \Delta u - e^{i(\gamma-\theta)} |u|^\alpha u - i\omega e^{-i\theta} u. \quad (5.2)$$

Note that F is well defined due to Sobolev embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{2\alpha+2}(\mathbb{R}^N)$.

Then $\varphi_\gamma = e^{i\omega_\gamma t} u_\gamma$ is a solution of (1.2) if and only if $F(\gamma, \omega_\gamma, u_\gamma) = 0$. Note that $F(\theta, 0, Ue^{i\beta}) = F(\theta, g(\theta)) = 0$. In addition, it is immediate to see that F is a C^1 function such that

$$\begin{aligned}\frac{\partial F}{\partial \omega}(\gamma, \omega, u)\mu &= -ie^{-i\theta}u\mu, \\ \frac{\partial F}{\partial u}(\gamma, \omega, u)v &= \rho v - \Delta v - e^{i(\gamma-\theta)}[|u|^\alpha v + \alpha|u|^{\alpha-2}u\Re(\bar{u}v)] - i\omega e^{-i\theta}v.\end{aligned}$$

By the surjective form of the Implicit Function Theorem [37, Theorem 4.H, p.177], the proof will be completed once we show that $\partial_{\omega,u}F(\theta, 0, Ue^{i\beta}) : X \rightarrow L^2(\Omega)$ is surjective. Note that

$$\frac{\partial F}{\partial \omega}(\theta, 0, Ue^{i\beta}) = -ie^{-i\theta}Ue^{i\beta}, \quad (5.3)$$

$$\frac{\partial F}{\partial u}(\theta, 0, Ue^{i\beta})v = \rho v - \Delta v - U^\alpha v - \alpha U^\alpha e^{i\beta}\Re(e^{-i\beta}v), \quad (5.4)$$

so that

$$\partial_{\omega,u}F(\theta, 0, Ue^{i\beta})(\mu, v) = e^{i\beta}\partial_{\omega,u}F(\theta, 0, U)(\mu, e^{-i\beta}v).$$

It thus suffices to consider the case $\beta = 0$.

Given $f \in L^2_{\text{rad}}(\Omega)$, $\partial_{\omega,u}F(\theta, 0, U)(\mu, v) = f$ is equivalent to

$$-ie^{-i\theta}U\mu + Lv = f, \quad (5.5)$$

where L is given by (2.6). Note that L is a self-adjoint operator in $L^2_{\text{rad}}(\mathbb{R}^N)$ for the scalar product (5.1). Using that $\ker L = [iU]$, see Proposition 3.4, we choose μ such that

$$\tilde{f} = f + ie^{-i\theta}U\mu \in (iU)^\perp,$$

i.e.,

$$\mu = -\frac{1}{\cos \theta \|U\|_{L^2(\mathbb{R}^N)}^2} \int_{\mathbb{R}^N} \Im f U. \quad (5.6)$$

The fact that $Lv = \tilde{f}$ has a solution for $\tilde{f} \in (iU)^\perp$ follows from the Fredholm Alternative applied to the compact operator $K = (\rho - \Delta)^{-1}U^\alpha$, see Section 3. This shows that L is surjective and closes the proof. \square

Proof of Theorem 1.2. Set $X = \mathbb{R} \times (H^2(\Omega) \cap H_0^1(\Omega))$ and define $F : (-\pi/2, \pi/2) \times X \rightarrow L^2(\Omega)$ by (5.2). The arguments of the proof of Theorem 1.1 are still valid in this case where Ω is a ball. The fact that $\ker L = [iU]$ was established in Proposition 4.6. \square

Remark 5.1. (1) Let Ω be the unitary ball of \mathbb{R}^N and let $\tilde{X} = \mathbb{R} \times H^2(\Omega) \cap H_0^1(\Omega) \cap (iU)^\perp$. Given $\tilde{f} \in (iU)^\perp$, there exists a unique $\tilde{z} \in (iU)^\perp$ such that $L\tilde{z} = \tilde{f}$. We may thus modify the proof of Theorem 1.2 and apply the standard Implicit Function Theorem to find a unique curve of $g_\gamma = (\omega_\gamma, u_\gamma)$ in \tilde{X} such that $\varphi_\gamma = e^{i\omega_\gamma t} u_\gamma$ is a standing wave solution of (1.2). Since the equation is invariant under unitary transformations and U is radially symmetric, it follows by the uniqueness of g_γ that φ_γ is radially symmetric.

(2) Theorem 1.2 is still valid in the case $N = 2$, Ω any bounded regular star shaped domain and $\rho = 0$. From Remark 4.4 the argument of the proof applies without any change.

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